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Pascal Morin and Claude Samson

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Abstract: Rigid body models with two controls cannot be locally asymptotically stabilized by continuous feedbacks which are functions of the state only. However, this impossibility does no longer hold when the feedback is also a function of time, and explicit smooth time-varying feedbacks which locally asymptotically stabilize the attitude of a rigid spacecraft have previously been proposed by the authors and colleagues [10]. Due to the smoothness of the control law, the stabilization is not exponential and the asymptotical convergence rate to the desired equilibrium is only polynomial in the worst case. Nevertheless, exponential convergence can be obtained by considering time-varying feedbacks which are only continuous at the equilibrium. The present article proposes explicit feedback control laws of this type.

Key-words: Time-varying control, attitude stabilization, homogeneous system, continuous feedback.

(Résumé : tsvp)

Unité de recherche INRIA Sophia-Antipolis
2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex (France)
Téléphone : (33) 93 65 77 77 – Télécopie : (33) 93 65 77 65

Stabilisation Exponentielle Instationnaire d'un Satellite avec Deux Commandes

Résumé : L'orientation d'un corps rigide auquel sont appliquées deux commandes ne peut être stabilisée par retour d'état continu fonction seulement de l'état. La stabilisation par retour d'état continu dépendant également du temps reste cependant possible, et de tels retours d'état, infiniment différentiables et localement asymptotiquement stabilisants, ont été proposés par les auteurs et leurs collègues dans [10]. En raison de la régularité de ces lois de commande, le taux de convergence asymptotique vers l'équilibre désiré est, dans le plus mauvais cas, seulement polynômial. Toutefois, une convergence exponentielle peut être obtenue en considérant des retours d'état instationnaires seulement continus. De tels retours d'état sont proposés dans le présent rapport.

Mots-clé : Commande instationnaire, stabilisation d'attitude, système homogène, retour d'état continu.

1 Introduction

Following an idea which may be traced back to a work by Sontag and Sussmann [16] in 1980, an article by Samson [15] in 1990 has revealed that continuous time-varying feedbacks, i.e. feedbacks which depend not only on the system's state vector but also on time, can be of interest to stabilize many systems which cannot be stabilized by continuous pure-state feedbacks. This has been confirmed by Coron's results [3] which establish that most STLC (Small Time Locally Controllable) systems can be stabilized by continuous time-varying feedback.

It is known that a given attitude for a rigid spacecraft with only two controls cannot be asymptotically stabilized by means of continuous state-feedbacks, as pointed out for example in [1], since Brockett's necessary condition [2] for smooth feedback stabilizability is not satisfied in this case. Nevertheless, existence of stabilizing continuous time-varying feedbacks for this problem follows from [3] and [8], with the latter reference establishing that the system is STLC.

In [10], explicit smooth time-varying feedbacks have been derived by using center manifold theory, time-averaging and Lyapunov techniques. However, due to the smoothness of the control law, the asymptotical rate of convergence to zero of the closed-loop system's solutions is only polynomial in the worst case. In the present paper, we derive time-varying continuous feedbacks which locally *exponentially* asymptotically stabilize the attitude of a rigid spacecraft. Our construction relies on the properties of homogeneous systems, combined with averaging and Lyapunov techniques. It also uses a specific *cascaded high-gain control* result established here for systems homogeneous of degree zero involving controls which are not necessarily differentiable everywhere.

Another solution, yielding also exponential stabilization, has recently been proposed by Coron and Kerai in [4]. A particularity of this solution is that it consists of switching periodically between two control laws, one of which depends on time. The resulting feedback control is continuous and time-periodic. By contrast, the solution proposed here consists of a single and simpler control expression.

The paper is organized as follows. In Section 2, the equations of a rigid body, when using a set of Rodrigues parameters to represent attitude errors, are recalled and the control objective is stated. In section 3, general properties and stabilization results associated with homogeneous systems, which are useful to establishing our main result, are recalled. The aforementioned cascaded high-gain control result, for systems which are homogeneous of degree zero, is derived in Section 4. A set of continous time-varying control laws which locally asymptotically and exponentially stabilizes the desired attitude is proposed in Section 5, with the corresponding proof

of stability. Finally, simulation results are given in section 6.
Throughout the paper, we use the following notations:

- $|\cdot|$ denotes the Euclidean norm.
- $\langle .,. \rangle$ denotes the Euclidean cross product.
- I_3 denotes the identity matrix in $\mathbb{R}^3$.
- A function $f : \mathbb{R}^n \longmapsto \mathbb{R}^p$ is of class C^p (resp. C^∞) if it has continuous partial derivatives up to order p (resp. at any order).

2 Equations of the rigid body

Let us consider:

- a frame F_0 attached to the spacecraft and whose axes correspond to the principal inertia axes of the body.
- a fixed frame F_1 whose attitude is the desired one for F_0.

and denote:

- ω: the angular velocity vector of the frame F_0 with respect to the frame F_1, expressed in the basis of F_0.
- J: the (diagonal) inertia matrix:

$$J = \begin{pmatrix} j_1 & 0 & 0 \\ 0 & j_2 & 0 \\ 0 & 0 & j_3 \end{pmatrix} \tag{1}$$

- $S(\omega)$: the matrix representation of the cross product:

$$S(\omega) = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \tag{2}$$

If R is the rotation matrix representing the attitude of F_1 with respect to F_0 (and whose columns vectors are the basis vectors of F_1 expressed in F_0) , we get the well known equations:

$$\left\{ \begin{array}{rcl} \dot{R} & = & S(\omega)R \\ J\dot{\omega} & = & S(\omega)J\omega + B(\tau_1, \tau_2, 0)^T \end{array} \right. \tag{3}$$

where the τ_i are the torques applied to the rigid body and B represents the directions in which these torques are applied.
We make the assumption that $B = I_3$ (i.e. that the torques are applied in the direction of principal inertia axes). However, our result can be easily extended to any location of the actuators for which the spacecraft is controllable, after an adequate change of state and control variables similar to the one proposed in [4].

System (3) is a control system with two scalar inputs τ_1 and τ_2 and state space $SO(3) \times \mathbb{R}^3$. Our objective is to find a control $(\tau_1(t, R, \omega),\ \tau_2(t, R, \omega))$ periodic with respect to time, which locally exponentially stabilizes the point $(I_3, 0)$ of $SO(3) \times \mathbb{R}^3$.

In order to control the body rotations, a preliminary step traditionally consists in defining a minimal set of local coordinates for the parametrization of $SO(3)$ around I_3. As in [10], we choose a set of coordinates, sometimes called Rodrigues parameters. To any rotation R of angle $\theta \in]-\pi, \pi[$ and axis the direction of which is defined by the unit vector $\vec{u}$, we associate the following three-dimensional vector :

$$X \;=\; (x_1, x_2, x_3)^T \;=\; tan(\frac{\theta}{2})\ u$$

with u denoting the coordinates of the vector $\vec{u}$ in the frame of F_0. It is shown in [10] that the system (3) can be written in the coordinates (X, ω) :

$$\left\{ \begin{array}{rcl} \dot{X} & = & \frac{1}{2}(\ \omega \ + \ S(\omega)\,X \ + \ \langle\omega, X\rangle\, X\ ) \\ \dot{\omega}_1 & = & c_1\,\omega_2\omega_3 + u_1 \\ \dot{\omega}_2 & = & c_2\,\omega_1\omega_3 + u_2 \\ \dot{\omega}_3 & = & c_3\,\omega_1\omega_2 \end{array} \right. \tag{4}$$

with $c_1 = \dfrac{j_2 - j_3}{j_1}, c_2 = \dfrac{j_3 - j_1}{j_2}, c_3 = \dfrac{j_1 - j_2}{j_3}, u_1 = \dfrac{\tau_1}{j_1}$, and $u_2 = \dfrac{\tau_2}{j_2}$.

It is of course assumed that $c_3 \neq 0$, since otherwise the system would not be controllable nor stabilizable. Moreover we may also assume that $c_3 > 0$, due to the fact that the change of variables

$$(x_1, x_2, x_3, \omega_1, \omega_2, \omega_3, u_1, u_2) \longmapsto (x_2, x_1, -x_3, \omega_2, \omega_1, -\omega_3, u_2, u_1)$$

leaves the equation (4) unchanged, except for the parameters (c_1, c_2, c_3) which are changed into $(-c_2, -c_1, -c_3)$.

Our objective is to find a continuous feedback control law which exponentially asymptotically stabilizes the origin of (4).

3 Homogeneity and exponential stabilization

Let us first recall some results and definitions about homogeneous systems. For a more complete exposition, the reader is referred to [7] or [6].
For any $\lambda > 0$ and any set of real parameters $r_i > 0$ $(i = 1, \ldots, n)$, one defines the following "dilation" operator $\delta_\lambda^r : \mathbb{R}^n \longmapsto \mathbb{R}^n$ by

$$\delta_\lambda^r(x_1, \ldots, x_n) = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n).$$

An *homogeneous* norm associated with this dilation operator is:

$$\rho_p^r(x) = (\sum_{j=1}^{n} |x_j|^{\frac{p}{r_j}})^{\frac{1}{p}} \qquad with\ p > 0.$$

A continuous function $f : \mathbb{R}^n \longmapsto \mathbb{R}$ is homogeneous of degree $\tau \geq 0$ with respect to the dilation δ_λ^r if :

$$\forall \lambda > 0, \quad f(\delta_\lambda^r(x)) = \lambda^\tau f(x).$$

A differential system $\dot{x} = f(x)$ (or a vector field f), with $f : \mathbb{R}^n \longmapsto \mathbb{R}^n$ continuous, is homogeneous of degree $\tau \geq 0$ with respect to the dilation δ_λ^r if :

$$\forall \lambda > 0, \quad f_i(\delta_\lambda^r(x)) = \lambda^{\tau + r_i} f_i(x) \qquad (i = 1, \ldots, n).$$

The above definitions can be extended to time-dependant functions and systems. Such an extension has already been considered in [11] and simply follows by considering the extended dilation operator:

$$\delta_\lambda^r(x_1, \ldots, x_n, t) = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n, t).$$

The definitions remain unchanged.
The following result, which is a particular case of a proposition by Pomet and Samson, establishes the existence of homogeneous Lyapunov functions for time-varying

asymptotically stable systems which are homogeneous of degree zero with respect to some dilation. This proposition extends a theorem by Rosier [13] on autonomous systems.

Proposition 1 (Pomet, Samson [11]) *Let $f(x,t) : \mathbb{R}^n \times \mathbb{R} \longmapsto \mathbb{R}^n$ a T-periodic continuous function $(f(x,t+T) = f(x,t))$. Assume that the system:*

$$\dot{x} = f(x,t) \tag{5}$$

is homogeneous of degree 0 with respect to a dilation $\delta_\lambda^r(x,t)$ and that $x = 0$ is an asymptotically stable equilibrium of (5).
Then, for any $\alpha > 0$ and $p < \frac{\alpha}{max\{r_j\}}$, there exists a function $V(x,t) : \mathbb{R}^n \times \mathbb{R} \longmapsto \mathbb{R}$ such that:

- *V is of class C^p on $\mathbb{R}^n \times \mathbb{R}$ and of class C^∞ on $(\mathbb{R}^n - \{0\}) \times \mathbb{R}$.*
- *V is T-periodic $(V(x,t+T) = V(x,t))$.*
- *V est homogeneous of degree α with respect to the dilation δ_λ^r:*
$$V(\delta_\lambda^r(x,t)) = \lambda^\alpha V(x,t)$$
- *$V(x,t) > 0$ if $x \neq 0$, $V(0,t) = 0$.*
- *$V(x,t)$ is "proper" with respect to x:*
  $\forall t \ : \ V(x,t) \longmapsto +\infty$ when $|x| \longmapsto +\infty$.
- *$\exists M > 0, \ \exists \alpha > 0 \ : \ \frac{\partial V}{\partial t}(x,t) + \frac{\partial V}{\partial x}(x,t) \ f(x,t) \leq -M(\rho_p^r(x))^\alpha$.*

The following properties can be viewed as a consequence of the above proposition. The first property has been stated by Kawski in [7], in the case of autonomous systems. The second has been shown by Hermes, also for autonomous systems in which case no assumption on the homogeneity degree of the vector field is needed (see [6]).

Proposition 2 (exponential stabilization) *Consider the system*

$$\dot{x} = f(x,t) \tag{6}$$

with $f(x,t) : \mathbb{R}^n \times \mathbb{R} \longmapsto \mathbb{R}^n$ a T-periodic continuous function $(f(x,t+T) = f(x,t))$, and $f(0,t) = 0$. Assume that (6) is homogeneous of degree 0 with respect to a dilation $\delta_\lambda^r(x,t)$ and that the equilibrium point $x = 0$ of this system is locally asymptotically stable.
Then,

i) *$x = 0$ is globally exponentially stable* **in the sense** *that there exists two strictly positive constants K and γ such that along any solution of the system (6),*

$$\rho_p^r(x(t)) \leq K\, e^{-\gamma t} \rho_p^r(x(0))$$

with $\rho_p^r(x)$ denoting an homogeneous norm associated with the dilation $\delta_\lambda^r(x,t)$

ii) *the solution $x = 0$ of the "perturbed" system :*

$$\dot{x} = f(x,t) + g(x,t)$$

is locally exponentially stable when $g(x,t) : \mathbb{R}^n \times \mathbb{R} \longmapsto \mathbb{R}^n$ is a continuous T-periodic function such that the corresponding vector field g is a sum of homogeneous vector fields of degree strictly positive with respect to δ_λ^r.

For the proof of part *i)*, we refer to the proof of [11, Prop. 1]. The proof of part *ii)* follows from Proposition 1 and from the proof of [13, Th 3].

The next Proposition is a corollary of a result by M' Closkey and Murray.

Proposition 3 (M'Closkey, Murray [9]) *Consider the system :*

$$\dot{x} = f(x, t/\epsilon) \tag{7}$$

with $f(x,t) : \mathbb{R}^n \times \mathbb{R} \longmapsto \mathbb{R}^n$ a continuous T-periodic function $(f(x, t+T) = f(x,t))$. Assume that (7) is homogeneous of degree 0 with respect to a dilation $\delta_\lambda^r(x,t)$ and that the origin $y = 0$ of the " averaged system"

$$\dot{y} = \bar{f}(y) \tag{8}$$

(with $\bar{f}(y) = \dfrac{1}{T}\displaystyle\int_0^T f(y,t)\,dt$) is asymptotically stable.
Then, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, the origin $x = 0$ of (7) is exponentially stable.

The following proposition complements the previous one for a specific class of systems, in the sense that it provides us with a value of ϵ_0.

Proposition 4 *Consider the system :*

$$\dot{x} = f_0(x) + \sum_{i=1}^{p} g_i(t/\epsilon) f_i(x) \tag{9}$$

where $f_i(i = 0,\ldots,p) : \mathbb{R}^n \longmapsto \mathbb{R}^n$ are continuous functions which define homogeneous vector fields of degree 0 with respect to a dilation $\delta^r_\lambda(x)$, $f_i(i = 1,\ldots,p)$ are functions of class C^1 on $\mathbb{R}^n - 0$, and $g_i(i = 1,\ldots,p) : \mathbb{R} \longmapsto \mathbb{R}$ are continuous T-periodic functions such that $\int_0^T g_i(\tau)\,d\tau = 0$.
Assume that the origin $x = 0$ of the "averaged system" :

$$\dot{x} = f_0(x) \tag{10}$$

is asymptotically stable and that an associated Lyapunov function $V(x)$ of class C^2, homogeneous of degree β with respect to $\delta^r_\lambda(x)$, such that $V(x) \geq K_1(\rho^r_p(x))^\beta$ and $\frac{\partial V}{\partial x}(x)f_0(x) \leq -K_2(\rho^r_p(x))^\beta$ is known.
Define also :

$$C_i \;=\; \sup_{t\in\mathbb{R}} |g_i(t)|, \quad I_i \;=\; \sup_{t\in\mathbb{R}} |\int_0^t g_i(\tau)\,d\tau|$$

$$\delta_i \;=\; \sup_{\rho^r_p(x)=1} |\frac{\partial V}{\partial x}(x)f_i(x)|, \quad \gamma_{i,j} \;=\; \sup_{\rho^r_p(x)=1} |\frac{\partial}{\partial x}(\frac{\partial V}{\partial x}(x)f_i(x))f_j(x)|$$

$$\epsilon_0 \;=\; Max\left\{\frac{K_1}{\sum_{i=1}^p I_i\delta_i}, \frac{K_2}{\sum_{i=1}^p I_i(\gamma_{i,0} + \sum_{j=1}^p C_j\gamma_{i,j})}\right\}, \tag{11}$$

Then for any $\epsilon \in (0,\epsilon_0)$ the origin $x = 0$ of the system (9) is exponentially stable.

Proof
The proof relies on the construction of a Lyapunov function for the system (9).
Since the functions g_i and $\left(\int_0^t g_i(\tau)\,d\tau\right)$ are T-periodic continuous functions, the values C_i and $I_i(i = 1,\ldots,p)$ are well defined.
Let us consider the following continuous periodic function, homogeneous of degree β with respect to the dilation $\delta^r_\lambda(x,t)$:

$$W(x,t) \;=\; V(x) - \epsilon\sum_{i=1}^p\left(\int_0^{t/\epsilon} g_i(\tau)\,d\tau\right)\frac{\partial V}{\partial x}(x)f_i(x). \tag{12}$$

Then for ϵ smaller than ϵ_0, and using the fact that $|\frac{\partial V}{\partial x}(x)f_i(x)| \leq \delta_i(\rho^r_p(x))^\beta$, it is simple to verify that W is a positive function. Moreover this function is of class C^1

on $(\mathbb{R}^n - 0) \times \mathbb{R}$. The time derivative of W along any trajectory of the system (9) which does not pass through $x = 0$ is then given by :

$$\dot{W} \;=\; \frac{\partial V}{\partial x} f_0(x) - \epsilon \sum_{i=1}^{p} \left(\int_0^{t/\epsilon} g_i(\tau)\, d\tau \right) \; \frac{\partial}{\partial x}(\frac{\partial V}{\partial x}(x) f_i(x)).(f_0(x) + \sum_{j=1}^{p} g_j(t/\epsilon) f_j(x)). \tag{13}$$

For ϵ smaller than ϵ_0, it is simple to verify that :

$$\dot{W} \leq -K(\rho_p^r(x))^{\beta}$$

with K a strictly positive constant.
(end of proof).

4 Cascaded high-gain control for a class of homogeneous systems

The next proposition concerns the classical problem of "adding integrators". For autonomous systems, the existence of asymptotically stabilizing homogeneous feedbacks, for an homogeneous asymptotically stabilizable system to which an integrator has been added at the input level, has been proved in [5]. Some (non systematic) constructive methods have also been developed in [12] and [14]. The following result provides a simple solution to this problem for a class of homogeneous time-periodic systems.

Proposition 5 *Consider the following system :*

$$\dot{x} = f(x, v(x^1, t), t) \tag{14}$$

with $f(x, y, t) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}^n$ *a continuous* T*-periodic function,* $x^1 = (x_1, \ldots, x_m)$, $m \leq n$, *and* $v(x^1, t) : \mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R}$ *a continuous* T*-periodic function, differentiable with respect to* t*, of class* C^1 *on* $(\mathbb{R}^m - \{0\}) \times \mathbb{R}$*, homogeneous of degree* q *with respect to a dilation* $\delta_\lambda^r(x, t)$*.*
Assume further that the system (14) is homogeneous of degree 0 with respect to the dilation $\delta_\lambda^r(x, t)$ *and that the origin* $x = 0$ *of this system is asymptotically stable.*
Then, for k *positive and large enough, the origin* $(x = 0, y = 0)$ *of the system*

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x, y, t) \\ \dot{y} & = & -k(y - v(x^1, t)) \end{array} \right. \tag{15}$$

is asymptotically stable.

Proof

Let :

- $$\delta(x,t) = (\lambda^{r_1}x_1, \dots, \lambda^{r_n}x_n, t) \tag{16}$$

  denote the dilation with respect to which the system (14) is homogeneous of degree 0, and $\rho(x)$ an associated homogeneous norm.

- $$\delta_e(x,y,t) = (\lambda^{r_1}x_1, \dots, \lambda^{r_n}x_n, \lambda^q y, t) \tag{17}$$

  denote the dilation with respect to which the system (15) is homogeneous of degree 0.

Since the function $v(x^1,t)$ is, by assumption, of class C^1 on $(\mathbb{R}^m - \{0\}) \times \mathbb{R}$ and homogeneous of degree q with respect to the dilation $\delta(x,t)$, the function $v^r(x^1,t)$ is also of class C^1 on $(\mathbb{R}^m - \{0\}) \times \mathbb{R}$ and it is homogeneous of degree rq, for any positive integer r. Consequently, the function $\dfrac{\partial v^r}{\partial x_i}$ is homogeneous of degree $rq - r_i$, for $i = 1, \dots, m$. The integer r is here chosen such that $r > max\{\frac{r_i}{q}, \quad 1 \le i \le m\}$. In this case each partial derivative of v^r is homogeneous of strictly positive degree with respect to the dilation $\delta(x,t)$, and thus tends to zero as $|x^1|$ tends to zero. Therefore, v^r is at least of class C^1 on $\mathbb{R}^m \times \mathbb{R}$. In what follows, it is further assumed that r is odd.

We denote as $V(x,t)$ a T-periodic Lyapunov function for the system (14), homogeneous of degree $\beta = (r+1)q$ with respect to the dilation $\delta(x,t)$, and of class C^1. Such a function exists by application of Proposition 1. Following the “desingularisation method” proposed in [12], we consider the following function :

$$W(x,y,t) \; = \; \gamma\, V(x,t) + \phi(y,x^1,t) \tag{18}$$

with $\gamma > 0$ and :

$$\phi(y,x^1,t) \; = \; \int_{v(x^1,t)}^{y} s^r - v^r(x^1,t)\, ds. \tag{19}$$

In order to prove the proposition, we show below that W is a Lyapunov function for the system (15), when γ and k are large enough.

We first note that ϕ is positive, and equal to zero if and only if $y = v(x^1,t)$. This already implies that W is positive and vanishes only at $(x,y) = (0,0)$. It is also

proper with respect to (x,y) since $V(x,t)$ is, by assumption, proper with respect to x and $\phi(y,x^1,t)$, seen as a function of y when x and t are fixed, tends to infinity when $|y|$ tends to infinity.
Now, from (19), it is simple to verify that :

$$\phi(y,x^1,t) = \frac{y^{r+1}}{r+1} + r\frac{v^{r+1}}{r+1} - v^r y$$

Since v^r is of class C^1, one deduces from the above expression that $\phi(y,x^1,t)$ is also of class C^1.
Let us now calculate the time derivative $\dot{W}$ of W along any trajectory of the system (15). With a slight abuse in the notations, introduced here for the sake of legibility, we have :

$$\begin{aligned} \dot{W} &= \gamma\dot{V} + \dot{\phi} \\ &= \gamma\frac{\partial V}{\partial x}f(x,v(x^1,t),t) + \gamma\frac{\partial V}{\partial t} \\ &\quad +\gamma\frac{\partial V}{\partial x}(f(x,y,t) - f(x,v(x^1,t),t)) \\ &\quad +\frac{\partial \phi}{\partial y}\dot{y} + \frac{\partial \phi}{\partial x^1}f^1(x,y,t) + \frac{\partial \phi}{\partial t} \end{aligned} \tag{20}$$

with f^1 denoting the vector-function whose components are the m first components of f.
Since V is a homogeneous of degree β Lyapunov function for the system (14), there exists a strictly positive constant K_1 such that :

$$\frac{\partial V}{\partial x}(x,t)\ f(x,v(x^1,t),t) \ + \ \frac{\partial V}{\partial t}(x,t) \ \leq \ -2K_1(\rho(x))^\beta \tag{21}$$

We show next that there exists another positive constant K_2 such that :

$$\left|\frac{\partial V}{\partial x}(x,t)\ (f(x,y,t) - f(x,v(x^1,t),t))\right| \leq K_1\,(\rho(x))^\beta + K_2\,(y - v(x^1,t))^{r+1}. \tag{22}$$

To this purpose, let us consider the following set of functions :

$$G_p(x,y,t) \ = \ \frac{\left|\frac{\partial V}{\partial x}(x,t)\ (f(x,y,t) - f(x,v(x^1,t),t))\right|}{K_1\,(\rho(x))^\beta + p\,(y - v(x^1,t))^{r+1}} \tag{23}$$

indexed by the positive integer p .
G_p is a continuous T-periodic function, homogeneous of degree 0 with respect to

the dilation $\delta_e(x,y,t)$, and it is well defined for $(x,y) \neq (0,0)$. Time-periodicity of G_p allows to consider that time lives on the compact set $S^1 = \mathbb{R}/TZ$ instead of $\mathbb{R}$. Since G_p is homogeneous of degree zero, G_p reaches its maximum at some point (x_p, y_p, t_p) in $S \times S^1$, with S denoting the unit sphere in $\mathbb{R}^{n+1}$. By compacity of $S \times S^1$, one can extract a sub-sequence $(x_{p_l}, y_{p_l}, t_{p_l})$, $l \in \mathbb{N}$, which converges to some point $(\bar{x}, \bar{y}, \bar{t}) \in S \times S^1$. Let us distinguish the following two cases :

i) $\bar{y} = v(\bar{x}^1, \bar{t})$
By continuity of f and v, the numerator of $G_{p_l}(x_{p_l}, y_{p_l}, t_{p_l})$ tends to zero as l tends to $+\infty$, and for l large enough the denominator is greater than $\frac{K_1}{2}(\rho(\bar{x}))^\beta > 0$, using the fact that $\bar{x}$ cannot be equal to zero. Indeed, if $\bar{x}$ were equal to zero, then $\bar{y} = v(0, \bar{t})$ would also be equal to zero, contradicting the fact that $(\bar{x}, \bar{y})$ belongs to S. As a consequence, $G_{p_l}(x_{p_l}, y_{p_l}, t_{p_l})$ must be smaller than 1 for large enough values of l.

ii) $\bar{y} \neq v(\bar{x}^1, \bar{t})$
By continuity of f and v, the numerator of $G_{p_l}(x_{p_l}, y_{p_l}, t_{p_l})$ is bounded independently of l, and the denominator tends to $+\infty$ as l tends to $+\infty$. Therefore, $G_{p_l}(x_{p_l}, y_{p_l}, t_{p_l})$ tends to zero as l tends to $+\infty$.

We thus have proved the existence of an integer p for which $|G_p(x,y,t)| < 1$. By taking K_2 equal to this integer, the inequality (22) follows.
Let us now consider the term $\dfrac{\partial \phi}{\partial y}\,\dot{y}$ of (20). From (19) and (15), we have :

$$\frac{\partial \phi}{\partial y}\,\dot{y} \;=\; -k(y^r - v^r(x^1,t))\,(y - v(x^1,t)) \tag{24}$$

We show below the existence of a strictly positive constant α such that :

$$\frac{\partial \phi}{\partial y}\,\dot{y} \;\leq\; -k\alpha\,(y - v(x^1,t))^{r+1}. \tag{25}$$

To this purpose, let us consider the following function (with r odd) :

$$h(x) = \quad 2^{r-1}[(1+x)^r - x^r] - 1 \quad (\geq 0,\ \forall x) \tag{26}$$

the positivity of which is easily established. By taking $x = \dfrac{v(x^1,t)}{y - v(x^1,t)}$, one has :

$$\frac{2^{r-1}}{(y - v(x^1,t))^r}(y^r - v^r(x^1,t)) - 1 \geq 0 \tag{27}$$

Multiplying each mumber of (27) by $(y - v(x^1,t))^{r+1}$, one obtains, in view of (24), the desired inequality (25) with $\alpha = 2^{1-r}$.
Finally, we have for some value K_3 :

$$|\frac{\partial\phi}{\partial x^1}(y,x^1,t)\ f^1(x,y,t) + \frac{\partial\phi}{\partial t}(y,x^1,t)| \ \leq\ K_3\,(\ (\rho(x))^\beta + (y - v(x^1,t))^{r+1}\ ) \quad (28)$$

This inequality comes from that the function :

$$\frac{\frac{\partial\phi}{\partial x^1}(y,x^1,t)\ f^1(x,y,t) + \frac{\partial\phi}{\partial t}(y,x^1,t)}{(\rho(x))^\beta + (y - v(x^1,t))^{r+1}}$$

is homogeneous of degree zero with respect to the dilation $\delta_e(x,y,t)$, well defined outside $(x,y) = (0,0)$, and is thus bounded.

By using (20), (21), (22), (25), and (28), one obtains :

$$\begin{array}{rcl} \dot{W} & \leq & -2\gamma K_1\,(\rho(x))^\beta + \gamma K_1\,(\rho(x))^\beta + \gamma K_2\,(y - v(x^1,t))^{r+1} \\ & & +K_3\,(\rho(x))^\beta + K_3\,(y - v(x^1,t))^{r+1} - k\alpha(y - v(x^1,t))^{r+1}. \end{array} \quad (29)$$

For any $\gamma > K_3/K_1$, and any $k > \frac{\gamma K_2 + K_3}{\alpha}$, $\dot{W}$ is negative, and equal to zero if and only if $x = 0$ and $y = 0$.
(end of proof).

Proposition 5 can be used recursively for a multi-input system to which an integrator has been added at each input level. More precisely, one easily deduces the following corollary :

Corollary 1 *Consider the following system :*

$$\dot{x} = f(x, v(x,t), t) \quad (30)$$

with $f(x,y,t) : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} \mapsto \mathbb{R}^n$ *a continuous* T*-periodic function, and* $v(x,t)$: $\mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^p$ *a continuous* T*-periodic vector-function whose components* $v_1(x,t), \ldots$ $v_p(x,t)$ *are differentiable with respect to* t*, of class* C^1 *on* $(\mathbb{R}^n - \{0\}) \times \mathbb{R}$*, and homogeneous respectively of degree* $q_1, \ldots, q_p$ *with respect to a dilation* $\delta^r_\lambda(x,t)$*.*
Assume further that the system (30) is homogeneous of degree zero with respect to the dilation $\delta^r_\lambda(x,t)$ *and that the origin* $x = 0$ *of this system is asymptotically stable. Then, for positive and large enough values of* $k_1, \ldots k_p$*, the origin* $(x = 0, y = 0)$ *of*

the system

$$\left\{\begin{array}{lcl} \dot{x} & = & f(x,y,t) \\ \dot{y}_1 & = & -k_1(y_1 - v_1(x,t)) \\ \vdots & & \\ \dot{y}_p & = & -k_p(y_p - v_p(x,t)) \end{array}\right. \tag{31}$$

is asymptotically stable.

5 Exponential stabilization of the rigid spacecraft

Our main result, which gives explicit time-varying stabilizing feedbacks for the spacecraft, is stated next:

Theorem 1 *Consider the functions :*

$$\left\{\begin{array}{lcl} v_1(X,\omega_3,t) & = & -k_1\, x_1 - \rho(X,\omega_3)\, sin(t/\epsilon) \\ v_2(X,\omega_3,t) & = & -k_2\, x_2 + \dfrac{1}{\rho(X,\omega_3)}\,(x_3+\omega_3) sin(t/\epsilon) \end{array}\right. \tag{32}$$

with $\rho(X,\omega_3)$ *any homogeneous norm associated with the dilation* $\delta_\lambda^r(X,\omega_3,t) = (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^2\omega_3, t)$*, and the following time-varying continuous feedback :*

$$\left\{\begin{array}{lcl} u_1(X,\omega,t) & = & -k_3\,(\omega_1 - v_1(X,\omega_3,t)) \\ u_2(X,\omega,t) & = & -k_4\,(\omega_2 - v_2(X,\omega_3,t)) \end{array}\right. \tag{33}$$

Then, for any positive parameters k_1 *and* k_2 *there exists* $\epsilon_0 > 0$ *such that for any* $\epsilon \in (0, \epsilon_0]$ *and large enough parameters* $k_3 > 0$ *and* $k_4 > 0$*, the feedback (33) locally asymptotically and exponentially stabilizes the origin of the system(4).*

Proof

Let us consider the following dilation :

$$\delta_e^r(X,\omega,t) = (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda\omega_1, \lambda\omega_2, \lambda^2\omega_3, t). \tag{34}$$

The system (4)-(33) can be rewritten as :

$$\begin{pmatrix} \dot{X} \\ \dot{\omega} \end{pmatrix} \quad = \quad f(X,\omega,t) + g(X,\omega,t) \tag{35}$$

with

$$f(X,\omega,t) = (\frac{1}{2}\omega_1, \frac{1}{2}\omega_2, \frac{1}{2}(\omega_3+\omega_2 x_1-\omega_1 x_2), u_1(X,\omega,t), u_2(X,\omega,t), c_3\,\omega_1\omega_2)^T. \quad (36)$$

One easily verifies that $f(X,\omega,t)$ defines a continuous T-periodic vector field homogeneous of degree zero with respect to the dilation $\delta_e^r(X,\omega,t)$, and that $g(X,\omega,t)$ is continuous and defines a sum of homogeneous vector fields of degree strictly positive with respect to $\delta_e^r(X,\omega,t)$.
From Proposition 2, applied to (35), it is sufficient to show that the origin $(X = 0, \omega = 0)$ of the system :

$$\begin{pmatrix} \dot{X} \\ \dot{\omega} \end{pmatrix} = f(X,\omega,t) \quad (37)$$

is locally asymptotically stable.
To this purpose, let us first consider the following reduced system obtained from (37)-(36) by taking $v_1 \stackrel{\text{def}}{=} \omega_1$ and $v_2 \stackrel{\text{def}}{=} \omega_2$ as control variables :

$$\begin{array}{rcl} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} & = & \frac{1}{2}\begin{pmatrix} v_1 \\ v_2 \\ \omega_3 + v_2 x_1 - v_1 x_2 \end{pmatrix} \\ \dot{\omega}_3 & = & c_3\, v_1 v_2 \end{array} \quad (38)$$

With the controls v_1 and v_2 given by (32), one verifies, by application of Proposition 3, that the origin of the controlled system is asymptotically stable for any positive k_1 and k_2 and ϵ small enough.
Indeed, the vector-valued function associated with the right-hand side of the controlled system is continuous, since $v_1(X,\omega,t)$ and $v_2(X,\omega,t)$ are homogeneous of degree 1 with respect to the dilation $\delta_\lambda^r(X,\omega_3,t)$, are well defined outside the origin $(X = 0, \omega_3 = 0)$, and thus tend to zero as $|(X,\omega_3)|$ tends to zero. The corresponding vector field is also periodic and homogeneous of degree zero with respect to $\delta_\lambda^r(X,\omega_3,t)$, so that the assumptions of Proposition 3 are met. Moreover, the corresponding "averaged" system is given by :

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & -\frac{k_1}{2}\, x_1 \\ \dot{x}_2 & = & -\frac{k_2}{2}\, x_2 \\ \dot{x}_3 & = & \frac{1}{2}\,\omega_3 + \frac{1}{2}\,(k_1 - k_2)\, x_1 x_2 \\ \dot{\omega}_3 & = & c_3\,(k_1 k_2\, x_1 x_2 - \frac{1}{2}\, x_3 - \frac{1}{2}\,\omega_3). \end{array} \right. \quad (39)$$

and the origin of this system is locally asymptotically stable, since the linear approximation of this system around $x = 0$ is obviously stable.
The asymptotic stability of the origin of the system (37) follows by direct application of Corollary 1, after noticing that the functions $v_1(X, \omega_3, t)$ and $v_2(X, \omega_3, t)$ are of class C^1 on $(\mathbb{R}^3 \times \mathbb{R} - \{0, 0\}) \times \mathbb{R}$.
(end of proof).

6 Simulation results

The feedback laws given by (32)-(33) make the origin of the system (4) asymptotically stable for small enough values of ϵ and large enough values of k_3 and k_4. For practical purposes, it is necessary to specify values for which the stabilization is ensured. Conservative values can be determined via a complementary analysis. For instance, using the fact that $V(X, \omega_3) = 4x_1^4 + 4x_2^4 + x_3^2 + \omega_3^2 + x_3\omega_3$ is a Lyapunov function for the system (39) when $c_3 = k_1 = k_2 = 1$, one can deduce from Proposition 4 an upper bound for ϵ_0 . Conservative values of k_3 and k_4 can in turn be obtained by following the proof of Proposition 5. As for now, we will illustrate by simulation that ϵ does not have to be very small nor k_3 and k_4 very large. For example, the action of the control laws (32)-(33) on the system (4) has been simulated with the following choice of parameters :

$$\epsilon = 1/3, \quad k_1 = k_2 = 1, \quad k_3 = k_4 = 5.$$

with the initial conditions :

$$(x_1(0), x_2(0), x_3(0), \omega_1(0), \omega_2(0), \omega_3(0))^T = (0.5, 0.3, -1, 1, -1, 1)^T$$

The **figures 1-6** show the time evolution of the state variables $x_1, x_2, x_3, \omega_1, \omega_2, \omega_3$ and **figure 7** shows the linear decreasing of the *log* of the homogeneous norm used in the control laws, in order to illustrate the exponential convergence of this norm to zero.
It has also been verified by simulation that not any choice of the parameters yields stability.

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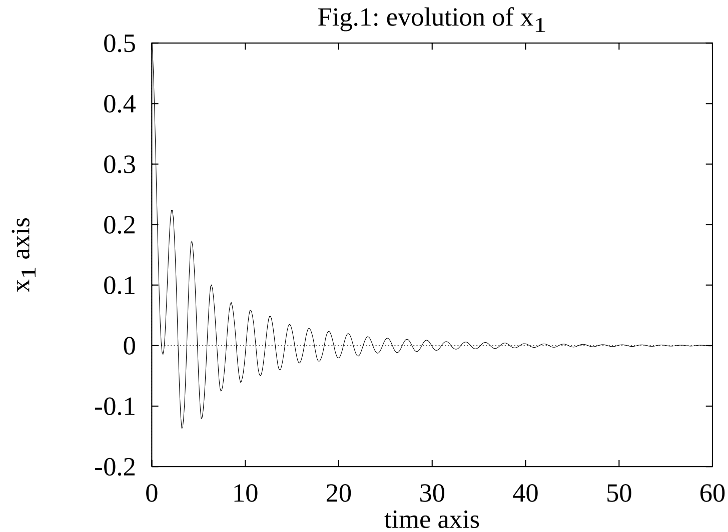
Fig.1: evolution of x1
x1 axis
time axis
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
0
10
20
30
40
50
60

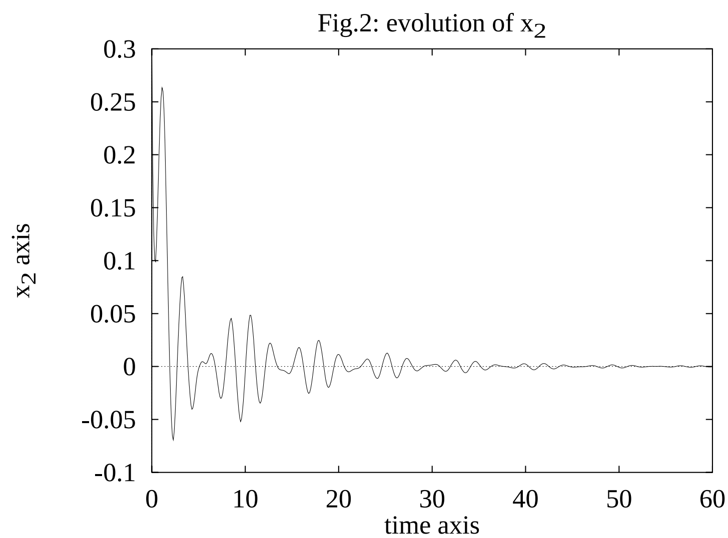
Fig.2: evolution of x2
x2 axis
time axis
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
0
10
20
30
40
50
60

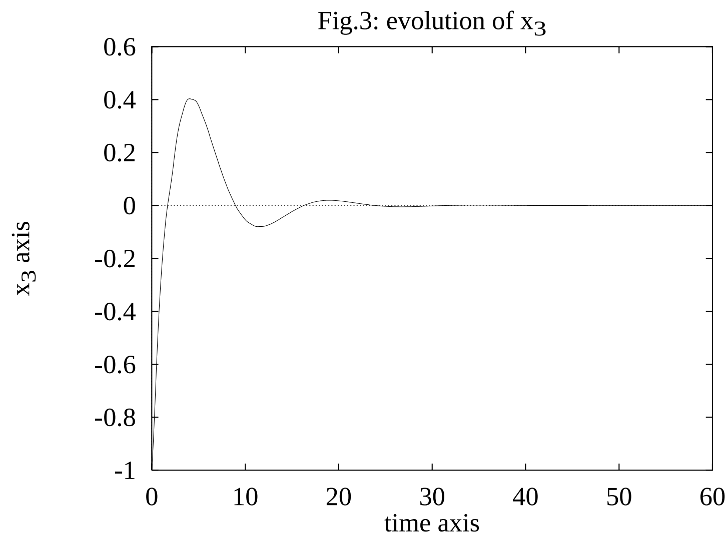
Fig.3: evolution of x_3
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
x_3 axis
0
10
20
30
40
50
60
time axis

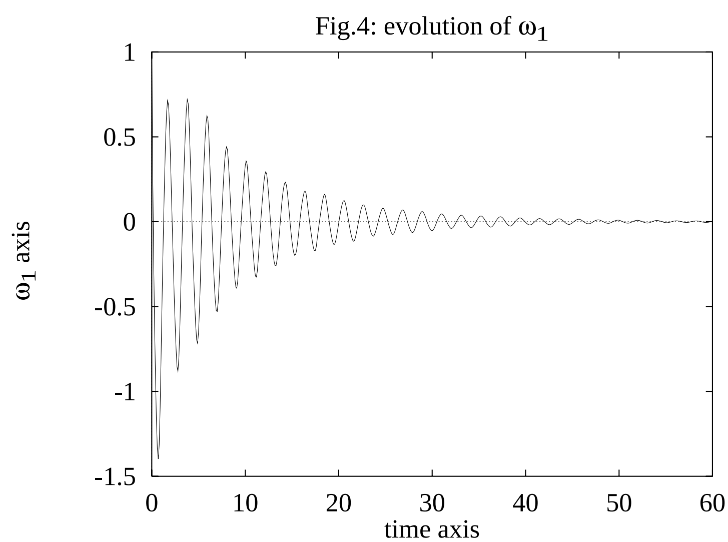
Fig.4: evolution of ω_1
1
0.5
0
-0.5
-1
-1.5
ω_1 axis
0
10
20
30
40
50
60
time axis

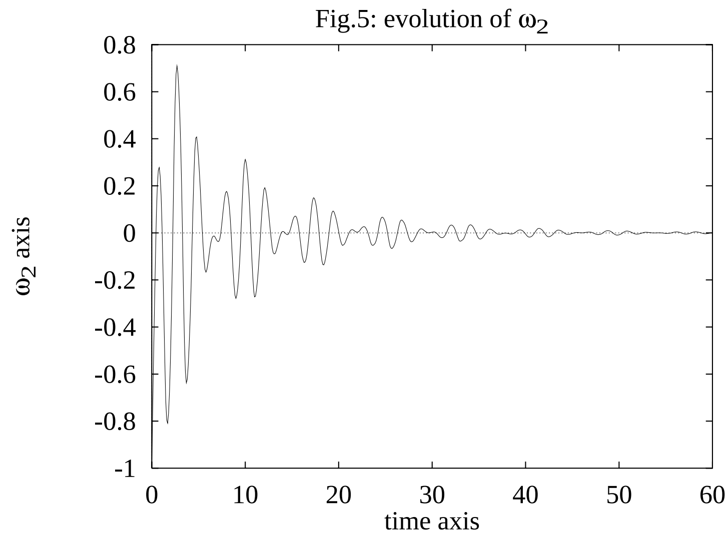
Fig.5: evolution of ω_2
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
10
20
30
40
50
60
ω_2 axis
time axis

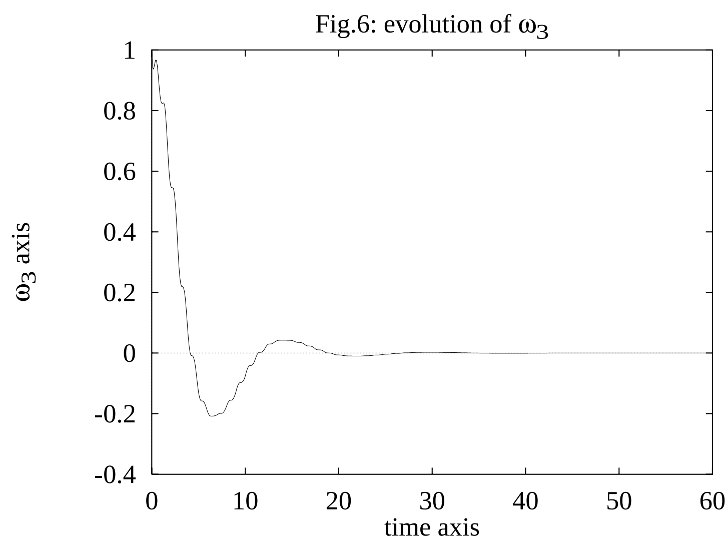
Fig.6: evolution of ω_3
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
10
20
30
40
50
60
ω_3 axis
time axis

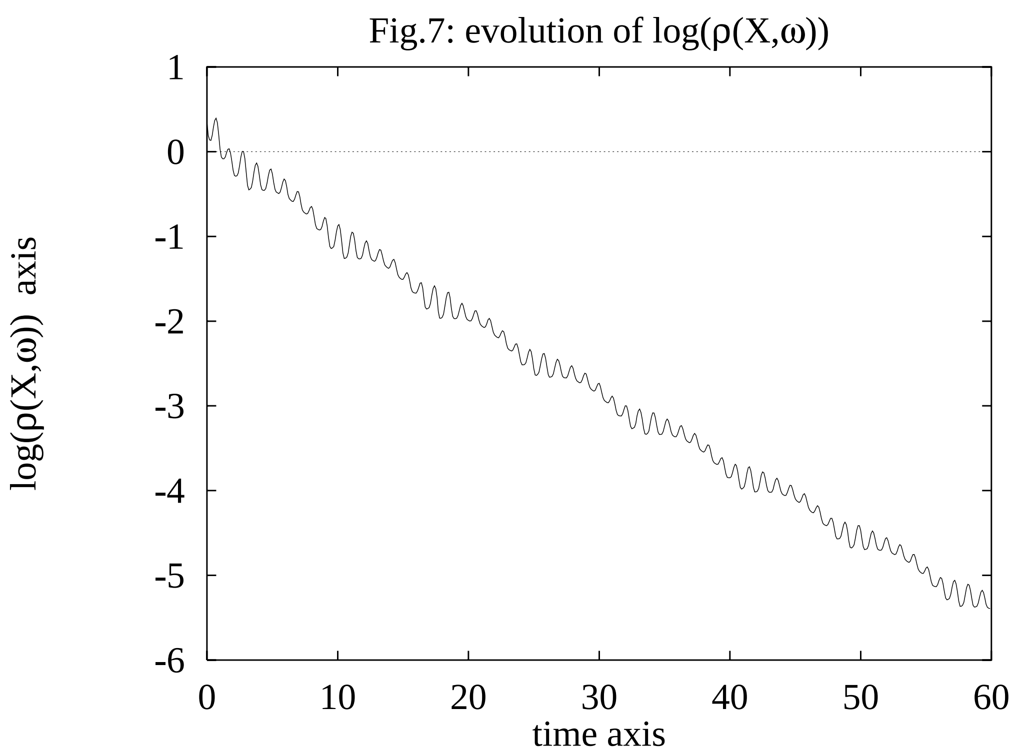
Fig.7: evolution of log(ρ(X,ω))
log(ρ(X,ω)) axis
1
0
-1
-2
-3
-4
-5
-6
0
10
20
30
40
50
60
time axis

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